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EGO-CENTERED AND LOCAL ROLES: A GRAPH THEORETIC APPROACH

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Structural equivalence (Lorrain and White, 1971) and automorphic equivalence (Everett, 1985) are generalized to define neighborhood- and ego-centered equivalences. It is shown that local versions of these equivalences can then be formulated quite naturally. In addition to these natural localizations, a generalized procedure capable of localizing any model of role equivalence is presented. From a theoretical point of view, local roles are recommended by the notion that network influences on ego diminish with distance. From a practical point of view, local roles help find structure in graphs where global equivalences find no two actors equivalent.

KEY WORDS: Network, graph, role, ego, local, equivalence.

1. INTRODUCTION

Mathematical formalizations of roles and positions in social networks have recently received a great deal of attention in the literature. Structural equivalence (Lorrain and White, 1971) was the seed for a variety of ideas of role equivalence. The concept of structural equivalence was relaxed by White, Boorman, and Breiger (1976) so as to allow an analysis of real data. The resultant concept was one of blockmodels—these were analyzed using blockmodel algebras or role structures by Boorman and White (1976).

Concepts of local roles as presented by a variety of researchers tended to be a generalization of these blockmodel algebras (Winship and Mandel, 1983; Wu, 1983;

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Mandel, 1983; Breiger and Pattison, 1986). In this paper we return to the original work of Lorrain and White (1971) and cast the definition of structural equivalence in graph theoretic terms. This will lead to a natural generalization of role equivalence that is similar, but with a subtle difference, to the one proposed by Winship and Mandel (1983) and Everett (1985). This new formalization is such that local ego-centered role equivalences can be formulated with minor modifications to the definitions. We shall also consider an alternative definition of local equivalence that in essence examines localized networks. This will allow other definitions of role equivalence to have a local form; for example we shall derive a local definition of regular equivalence (White and Reitz, 1983).

2. GRAPH THEORETIC DEFINITIONS OF ROLE EQUIVALENCE

Lorrain and White (1971) describe structural equivalence in terms of category theory. It is relatively simple to cast the definition in graph theoretic terms. The only aspect of their theory that is lost is the ability to use the "functorial reductions" or "semigroup equations" that is the most interesting but also the most ignored aspect of their theory (Boyd, 1983). The definitions presented here are applicable to graphs, digraphs, networks, as well as any of the "functorially reduced" versions of the original relations.

Suppose G is a labelled graph with vertex set V and edge set E. Then two vertices $a,b \in V$ are *structurally equivalent* iff the permutation $(a\ b)$ produces an automorphism of G. In other words, if we start from a labelled graph G and produce a new labelled graph G' by interchanging the labels of a and b, then a and b are structurally equivalent iff G and G' are isomorphic. This process is illustrated by the example shown in Figure 1, in which we demonstrate the structural equivalence of a and b. Note that c and d are also structurally equivalent. On the other hand, we see from Figure 2 that e and f are *not* structurally equivalent since the permutation $(e\ f)$ produces a graph G'' that is not isomorphic to G. This can easily be seen by noting that in the original graph G the point e is connected to both a and b, whereas in G'' it is not connected to either.

A natural generalization of structural equivalence as formulated above is to increase the number of vertices that can be permuted. For example, consider the following permutation for the graph in Figure 1: $(a\ c)(b\ d)(e\ f)$. This permutation produces the graph G''' shown in Figure 3. It is easily seen that G''' is isomorphic to G, hence we assume that a and c, b and d, and e and f are all role equivalent pairs. The permutation $(a\ b)(c\ d)$ shows further that there are two sets of role equivalent actors, namely $\{a,b,c,d\}$ and $\{e,f\}$. This generalization was first proposed by Winship and Mandel (1983) and, independently by Everett (1985) (see also Boyd and Everett (1988) and Everett and Borgatti (1988)).

In general, an *automorphism* of a graph G is a permutation π of the vertices V that has the property that $(a,b) \in E$ iff $(\pi(a),\pi(b)) \in E$. Two vertices, $a,b \in V$, belong to the same *orbit* of G iff $\pi(a) = b$ for some automorphism π. The set of orbits of a graph represents a partition into role equivalent actors. If two vertices belong to the same orbit, we shall say they are *automorphically equivalent*.

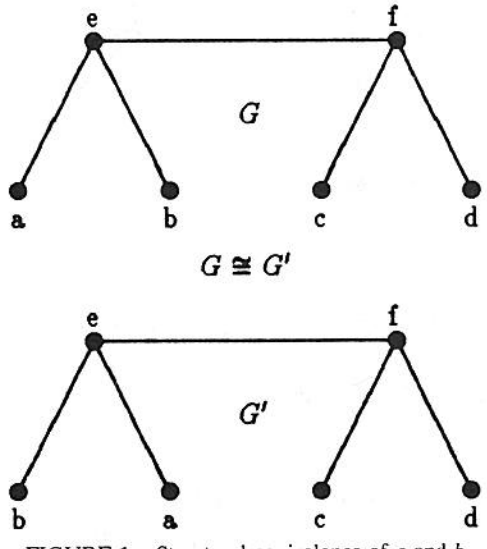

FIGURE 1. Structural equivalence of a and b.

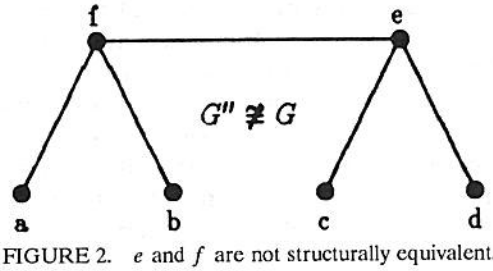

FIGURE 2. e and f are not structurally equivalent.

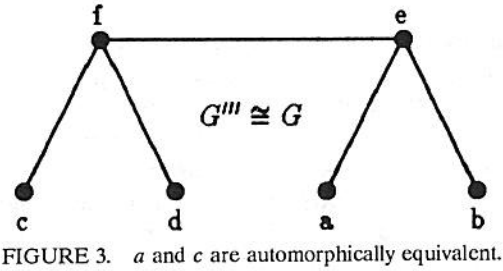

FIGURE 3. a and c are automorphically equivalent.

We shall now consider an alternative graph theoretic formulation of structural equivalence. If $S \subset V$ then the *neighborhood* $N(S)$ is defined by

$$N(S) = S \cup \{v \in V - S : (v,s) \in E,\ s \in S\}.$$

We define the higher-order neighborhoods inductively as

$$N^{i+1} = N(N^i(S)).$$

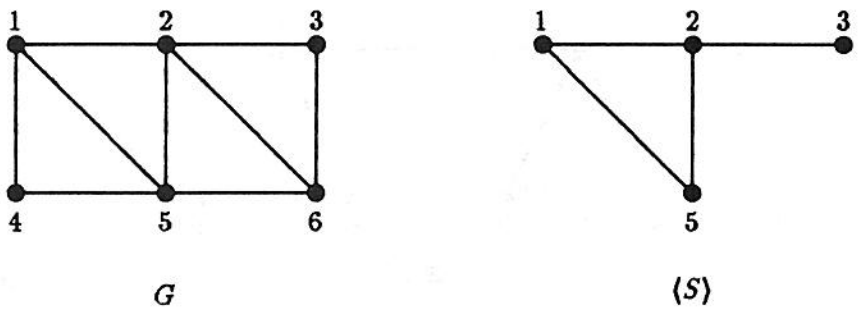

FIGURE 4. A graph G and the subgraph induced by the set $S = \{1,2,3,5\}$.

Hence in G of Figure 1 if $S = \{a\}$, then

$$N(S) = \{a,e\},$$
$$N^2(S) = \{a,b,e,f\},$$
$$N^3(S) = \{a,b,c,d,e,f\} = V.$$

If $S \subset V$ then $\langle S \rangle$ is the subgraph *induced* by S, i.e., the maximal subgraph of G with vertex set S. For example, Figure 4 shows $\langle S \rangle$ where $S = \{1,2,3,5\}$. If G is a graph (without self loops), then it is obvious that two vertices $a,b \in V$ are structurally equivalent iff $N(a) - \{a,b\} = N(b) - \{a,b\}$; i.e., aside from each other, they have precisely the same neighborhoods. Returning to the graph in Figure 1, we see immediately that $N(a) - \{a,b\} = N(b) - \{a,b\} = \{e\}$, showing again that a and b are structurally equivalent. If two vertices have identical neighborhoods, then it follows that all higher order neighborhoods are the same (again, we remove the vertices in question). A consequence of this observation is that for structurally equivalent vertices, a and b, the transposition $\pi = (a\ b)$ induces automorphisms:

$$\langle N^i(a) \rangle \cong \langle N^i(b) \rangle \qquad i = 1,2,\ldots,d(G) - 1. \tag{1}$$

The principle is basically the same in graphs with self loops, provided that a has a self loop iff b has one.

Although structural equivalence is a *local property*, depending only upon an individual's immediate neighborhood, the strictness of the condition implies a global equivalence in terms of higher-order neighborhoods. We therefore propose that rather than *equality* of first order neighborhoods, it is the *isomorphism* of all neighborhoods that captures the idea of role equivalence. This generalization is very similar in spirit to the idea of orbits. We therefore define two nodes $a,b \in V$ as *neighborhood-isomorphic* iff they satisfy 1.

If we return yet again to the graph in Figure 1, it is easily seen that e and f are two non-structurally equivalent actors who are neighborhood-isomorphic. The various neighborhood graphs are shown in Figure 5. As can be seen from this figure, it is not necessary to take all values of i from 1 to $d(G) - 1$ since the whole graph may be generated for values of i less than $d(G) - 1$; this will depend on the centrality of the vertices in question. Neighborhood-isomorphism is obviously an equivalence relation and consequently partitions the vertices into distinct equivalence classes.

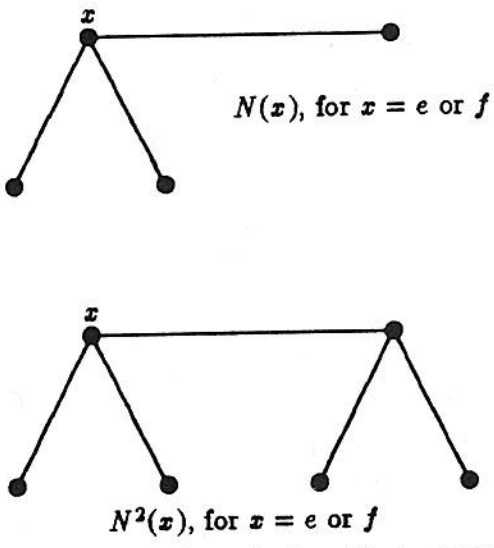

FIGURE 5. The first and second order neighborhoods of e and f.

The difference between neighborhood-centered and automorphic equivalence is subtle, and the reader may expect (as the authors originally did) that they are equivalent. The relationship between the definitions is explored in the following theorem and example.

THEOREM 1. *Two automorphically equivalent nodes are neighborhood-isomorphic.*

PROOF. The result is trivial when it is noted that if two nodes are automorphically equivalent, then they possess exactly the same properties. Hence they have the same degree, centrality, etc.; in addition they are connected to vertices with the same properties. In an unlabelled graph two automorphically equivalent nodes would be indistinguishable. Consequently, all their neighborhoods would be isomorphic. □

The converse of the above theorem is false, due to the following counter-example kindly pointed out to the authors by Norman Biggs. The (3,12) cage is a regular graph of girth 12 in which each vertex has degree 3. In addition, the graph has 126 vertices. The existence of such a graph is well known and it can be constructed using properties of finite projective geometries. The graph has two orbits of size 63; if x and y are in different orbits, then $\langle N^i(x)\rangle \cong \langle N^i(y)\rangle$ for $i < d$ since both graphs are simply trees. Hence x and y are neighborhood-isomorphic but not automorphically equivalent. For more information on the (3,12) cage, see Biggs (1974: 154–164). The reason that neighborhood-isomorphism fails to be an automorphic equivalence is that the isomorphism need not map one ego onto the other. We shall say that $\tilde{N}(x)$ is a *distinguished neighborhood* if it is a neighborhood of x with the vertex x distinguished. Figure 6 shows $\tilde{N}(2)$ for the graph of Figure 4, the distinguished vertex being represented by an 'open' circle.

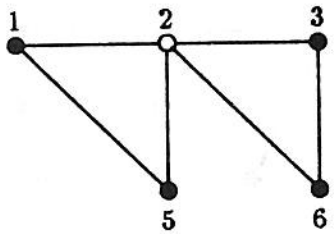

FIGURE 6. The distinguished neighborhood $\tilde{N}(2)$.

Two nodes $a, b \in V$ are *ego-isomorphic* iff they satisfy

$$\langle \tilde{N}^i(a) \rangle \cong \langle \tilde{N}^i(b) \rangle \qquad i = 1,2,\ldots,d(G)-1 \tag{2}$$

Since distinguished nodes must be mapped to distinguished nodes, when $i = d(G) - 1$ we must induce an automorphism of G. Hence we have

THEOREM 2. *Two nodes are ego-isomorphic iff they are automorphically equivalent.*

In all our previous examples, neighborhood- and ego-isomorphisms were equivalent. Obviously, ego-isomorphism implies neighborhood isomorphism, but the (3,12) cage shows that the converse is false. It seems likely that sophisticated mathematical examples will always be required to distinguish between these two concepts.

3. LOCAL ROLES

Ego-centered and neighborhood isomorphisms can easily be localized by simply restricting the order of the neighborhoods. We say that two nodes $a, b \in V$ are *k-locally ego isomorphic* iff

$$\langle \tilde{N}^i(x) \rangle \cong \langle \tilde{N}^i(y) \rangle \qquad i = 1,2,\ldots,k, \tag{3}$$

where $k < d(G) - 1$. If we look at the graph in Figure 7, we see that a and b are 1-locally ego isomorphic but are not 2-locally ego isomorphic. Hence a's role within his immediate contacts is exactly the same as b's. It is only when we look beyond the immediate contacts that we can distinguish between a and b. Local ego-centered roles capture the idea of individual's roles from their neighborhood perspective. Since it is assumed that influences further away from ego diminish due to attenuation within the network, it can be argued that local ego centered roles reflect social processes more accurately than other definitions.

In White and Reitz (1983) "regular" equivalence is defined as follows. If $G(V,R)$ is a graph and $\equiv$ is an equivalence relation on V, then $\equiv$ is a *regular equivalence* iff for all $a, a', b \in V$

$$[a \equiv a' \,\&\, aRb] \Rightarrow [(\exists b' \in V) b \equiv b' \,\&\, a'Rb']. \tag{4}$$

This definition can easily be extended to digraphs and sets of relations. Note that automorphic equivalence is regular. It is possible for digraphs to possess only one regular equivalence, namely, the trivial one in which each individual is placed in a

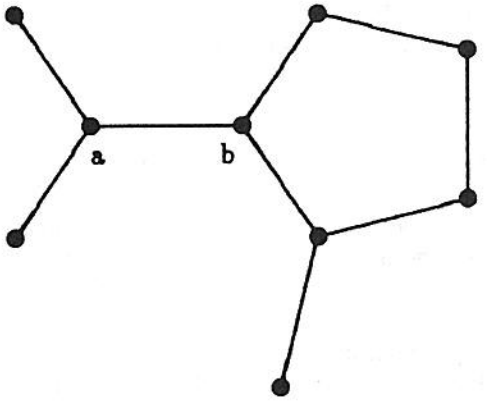

FIGURE 7. Points *a* and *b* are 1- but not 2-locally ego isomorphic.

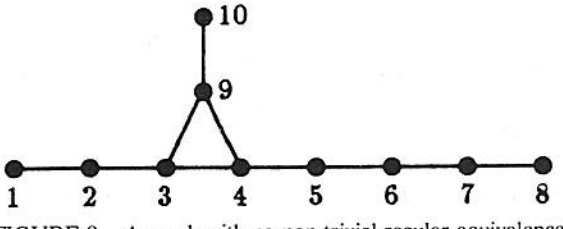

FIGURE 8. A graph with no non-trivial regular equivalence.

single equivalence class. In such graphs, local equivalences may be the only reasonable technique for extracting information on roles and positions from a particular data graph.

For undirected graphs the situation is slightly different, since in this case the partition that places every node in the same class is also regular. There may be regular partitions that lie between these extremes: Borgatti and Everett (1988) propose an algorithm that helps to find these equivalences. It is still possible, however, for a graph to have no non-trivial regular equivalences. The smallest graph that the authors have found with this property is given in Figure 8. It was shown that the graph contained no non-trivial regular equivalences by generating all 115,975 possible partitions and testing whether any of them satisfied condition 4 for regular equivalence. Whilst such an example is pathological and seems unlikely to occur in the analysis of real data, it does demonstrate in an extreme way the shortcomings of using a global analysis. It is simple to imagine situations in which a partition based upon regular equivalences does not give enough partitions to allow for any real analysis of data. Since every automorphic equivalence is also regular, this remark holds *a fortiori* for automorphic equivalence. It should be noted that for directed data, it is far more common for there to be only a trivial regular equivalence, which can become a serious problem in the analysis of real data.

Returning to the graph in Figure 8, we obtain the following partition based upon k-local ego isomorphism. A 1-local ego partition is given by the classes

$$\{1,8,10\}, \qquad \{2,5,6,7\}, \qquad \text{and} \qquad \{3,4,9\}.$$

At the 2-local level, we could only obtain two non-trivial groupings: $\{1,8\}$ and $\{3,4\}$.

The computation of ego centered equivalence requires the solution to the graph isomorphism problem. To determine whether two individuals are ego-isomorphic it is necessary to decide whether two graphs are isomorphic. This problem has received a lot of attention in the literature, but so far no polynomial time algorithm has been found for solving the general problem. However, there are efficient algorithms for solving the problem in certain special cases. For example, planar graphs and graphs of bounded degree can be tested for isomorphism in polynomial time (Hopcraft and Tarjan, 1973; Luks, 1982).

4. GENERAL LOCAL ROLES

As a practical matter, ego-isomorphic equivalence can be computed using the following procedure, which is based on the well-known trick of turning the problem of finding isomorphisms between graphs G_1 and G_2 into the problem of finding automorphisms α on the disjoint union, $G_1 \coprod G_2$, such that $\alpha(G_1) = G_2$. Recall that the *disjoint union* $A_1 \coprod A_2$ of two sets, A_1 and A_2, is just the ordinary union if the sets are already disjoint: if they are not disjoint, then two disjoint copies can be easily formed by "marking" each by taking Cartesian products with distinct singletons, so that

$$A_1 \coprod A_2 = (A_1 \times \{1\}) \cup (A_2 \times \{2\}),$$

although elements, (a,i), for $i = 1,2$, of the disjoint union are generally denoted more simply by subscripts, a_i. The *disjoint union* of two graphs, then, is just the disjoint union of their vertices and lines, respectively. If a and b are two vertices of a graph G, define

$$G^d_{ab} = \langle \tilde{N}^d(a) \rangle \coprod \langle \tilde{N}^d(b) \rangle,$$

where $d = d(G)$. The graph G^d_{ab} may then be submitted to an approximate automorphic equivalence program such as proposed by Everett and Borgatti (1988). Localization is then achieved by taking $d < d(G)$. Thus, if a and b are two nodes of G, then they are *k-locally ego-equivalent* iff they are in the same orbit of G^k_{ab}.

Note that localizing ego-equivalence means localizing automorphic equivalence. The advantage of using equation 4 is that it can be used to localize any definition of role-equivalence that can identify groups across components of a graph. Hence we can localize regular equivalence. Two actors, a and b, are *k-locally regularly equivalent* iff they are regularly equivalent in $\langle \tilde{N}^k(a) \rangle \coprod \langle \tilde{N}^k(b) \rangle$. For example, in Figure 9 the 1-locally regularly equivalent vertices are $\{a,b\}$, $\{c\}$, and $\{d,e,f\}$, which are not globally regularly equivalent, however. In fact, this graph contains no non-trivial regular equivalences.

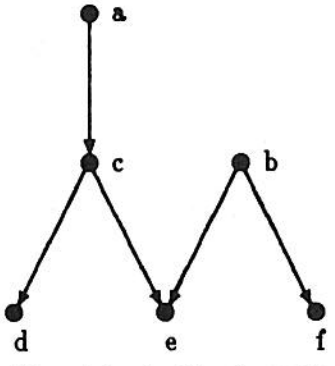

FIGURE 9. Digraph showing 1-locally regular equivalence.

A. ALGORITHM FOR GENERALIZED LOCALIZATION

Localizing a measure of role equivalence is straightforward. Programs localizing regular and automorphic equivalence are available as part of the NETPAC suite of network programs (Borgatti, 1988). In general terms, the algorithm for localizing any measure of role equivalence is as follows:

Set *Local Equiv*(i, j) = *TRUE* for all $i, j \in V$
For neighb $k = 1$ to d do
  For actor $i = 2$ to N do
  For actor $j = 1$ to $i - 1$ do
    if *Local Equiv*(i, j) = *TRUE* then do steps 1, 2, and 3:
      1. Create 2-component graph $G^k_{ab} = \langle \tilde{N}^k(i) \rangle \coprod \langle \tilde{N}^k(j) \rangle$, where the left component $\langle \tilde{N}^k(i) \rangle$ is the subgraph induced by the neighborhood of i, and the right component is $\langle \tilde{N}^k(j) \rangle$.
      2. Submit the graph to any equivalence procedure.
      3. If the copy of i in the left component turns out *NOT* equivalent to the copy of j in the right component, then set *Local Equiv*(i, j) = *FALSE*.

In the algorithm, d stands for the level of localization desired ($d = 1$ being maximum localization, $d = d(G)$ being global equivalence).

Note that the neighborhoods of i and j may contain points in common. In fact, i and j themselves may be found in each other's neighborhoods. If this occurs, it is important to remember that for i and j to be locally equivalent, they must be found equivalent *across* components, not within.

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